Linear System of Equations

A linear system of m equations in n unknowns $x_1, x_2, \dots x_n$ is a set of equations of the form,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \qquad (1)$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

The system is called linear because each variable x_j appear in the first power only just as in the equation of a straight line. $a_{11}, a_{12}, \dots a_{mn}$ are called coefficients of the system, $b_1, b_2 \dots b_m$ are also constants.

If all the b_j are zero then (1) is called homogeneous system. If at least one b_j is not zero then (1) is called a nonhomogeneous system.

Solution:

Solution of (1) is a set of numbers $x_1, x_2, \dots x_n$ that satisfies all the m equations.

Vectors:

A vector is a matrix with only one row or column. Its entries are called components of the vector.

a row vector $A = [a_1 \ a_2 \ \dots \ a_n]$

a column vector $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Solution vector. Is a vector x whose components form a solution of (1).

Trivial soln: A solution for which $x_1 = 0, x_2 = 0 \dots x_n = 0$ is called a trivial solution.

If the system (1) is homogeneous it has at least the trivial solution.

Matrix form of the Linear System (1)

The linear system of equations (1) may be written as a single vector equation,

$$AX = B$$

where $A = [a_{jk}]$ is the coefficient matrix, X is the variable matrix, B is the constant matrix given by,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & \dots & - \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented Matrix

The matrix $\tilde{A} = [A : B]$ formed by writing the vector B as a column vector to the right of A is called Augmented matrix given by,

$$\tilde{A} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Consistent system

A system of equations are consistent if it has at least one solution. If it has no solution system is inconsistent.

Elementary row operations:

There are three elementary row operations on a matrix,

(i) Interchange of two rows

(ii) Addition of a constant multiple of one row to another row.

(iii) Multiplication of a row by a nonzero constant c.

Note: The three row transformation will not alter the solution.

Row equivalent system

A linear system S_1 is row equivalent to a linear system S_2 if S_1 is obtained from S_2 by finitely many row operations. Row equivalent linear systems have the same set of solutions.

A linear system is called overdetermined if it has more equations than unknowns, $(m>n)$ determined if $m=n$ and underdetermined if it has fewer equations than unknowns $(m<n)$

Row Echelon form

The row echelon form of a matrix has the following characteristics,

(i) rows of zeros if present are the last rows.

(ii) in each nonzero row the left most nonzero entry is farther to the right than in the previous row.

Rank of the Matrix.

The numbes of nonzero rows in the Echelon form is called rank of the matrix.

Fundamental Theorem for linear system:

(i) If the rank of augmented matrix not equal to rank of coefficient matrix the system is inconsistent, i.e. no solution.

(ii) If rank of augmented matrix = rank of coefficient matrix & is equal to no. of unknowns then the system has unique solution

(iii) If the rank of augmented matrix is equal to rank of coefficient matrix and is less than no. of unknowns then the system has infinite no. of solutions.

Gauss Elimination and Back Substitution

1 Solve the linear system.

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

The augmented matrix is given by

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

First row is pivot row and coefficient 1 of first variable x_1 is called pivot. Now using this reduce,

$$\tilde{A} \approx \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \\ R_4 - 20R_1 \end{array}$$

interchanging R_2 & R_4

$$\tilde{A} \approx \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 - 3R_2$

$$\rightarrow \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now rank of $\tilde{A}$ same as rank of coefficient matrix, so it has a unique solution

By Back substitution,

we have, $-95x_3 = -190$

$x_3 = 2$

$10x_2 + 25x_3 = 90$

$10x_2 = 90 - 50$

$= 40$

$x_2 = 4$

$x_1 - x_2 + x_3 = 0$

$x_1 = x_2 - x_3$

$= 4 - 2$

$= 2$

Soln is, $x_1 = 2$, $x_2 = 4$, $x_3 = 2$

2) Solve

$$3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

Augmented matrix,

$$\tilde{A} = \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$R_2 - .6/3\, R_1$ & $R_3 - \frac{12}{3} R_1$

$$\tilde{A} = \left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$R_3 + R_2 \rightarrow$

$$\left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r = 2 < 3 \rightarrow$ infinite no. of solution.

by back substitution,

$$1.1\, x_2 + 1.1\, x_3 - 4.4\, x_4 = 1.1$$

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

i.e., $x_2 + x_3 - 4x_4 = 1$

i.e., $x_2 = 1 - x_3 + 4x_4$

also, $3x_1 = 8 - 2x_2 - 2x_3 + 54_4$

$= 8 - 2\left[1 - x_3 + 4x_4\right] - 2x_3 + 5x_4$

$= 8 - 2 + 2x_3 - 8x_4 - 2x_3 + 5x_4$

$= 6 - 3x_4$

i, $x_1 = 2 - x_4$

if $x_3 = t_1$ & $x_4 = t_2$

$x_1 = 2 - t_2$, & $x_2 = 1 - t_1 + 4t_2$

Hence there are infinite number of solution.

(3) $3x_1 + 2x_2 + x_3 = 3$
$2x_1 + x_2 + x_3 = 0$
$6x_1 + 2x_2 + 4x_3 = 6$

$$\tilde{A} = \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$R_2 - \frac{2}{3}R_1$ & $R_3 - 2R_1$

$$\rightarrow \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$R_3 - 6R_2$

$$\rightarrow \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Here we get the false statement $0=12$, which implies there is no solution. Or rank of A is not equal to rank of $\tilde{A}$. Hence solution is inconsistent.

Exercise

HW.

$$-3x + 8y = 5$$
$$8x - 12y = -11$$

$$\tilde{A} = \left[\begin{array}{cc:c} -3 & 8 & 5 \\ 8 & -12 & -11 \end{array}\right]$$

$R_2 + \frac{8}{3} R_1$

$$\rightarrow \left[\begin{array}{cc:c} -3 & 8 & 5 \\ 0 & \frac{28}{3} & +7/3 \end{array}\right]$$

$-12 + \frac{8\times 8}{3}$

$\frac{-36+64}{3}$

$\frac{28}{3}$

$$\frac{28}{3} x_2 = +\frac{7}{3}$$
$$x_2 = \frac{+7}{28} = \frac{+1}{4}$$

$$-3x_1 + 8x_2 = 5$$
$$-3x_1 = 5 - 8x_2 = 5 - 8 \times \frac{+1}{4}$$
$$= 5 - 2 = 3$$
$$x_1 = -1$$

(2) $8y + 6z = -4$

$-2x + 4y - 6z = 18$

$x + y + -z = 2$

$$\tilde{A} = \left[\begin{array}{ccc:c} 0 & 8 & 6 & -4 \\ -2 & 4 & -6 & 18 \\ 1 & 1 & -1 & 2 \end{array}\right]$$

R_1 & R_3 interchanged.

$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ -2 & 4 & -6 & 18 \\ 0 & 8 & 6 & -4 \end{array}\right]$$

$R_2 + 2R_1$

$$\rightarrow \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 6 & -8 & 22 \\ 0 & 8 & 6 & -4 \end{array}\right]$$

$R_3 - 6 \times \frac{8}{6}$ ie, $R_3 - 6 \times \frac{4}{3}$

$$\rightarrow \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 6 & -8 & 22 \\ 0 & 0 & \frac{50}{3} & \frac{-100}{3} \end{array}\right]$$

Back substitution,

$\frac{50}{3} z = \frac{-100}{3} \Rightarrow z = -2$

$6y - 8z = 22 \Rightarrow 6y = 22 + 8z$

$= 22 + 8 \times -2 = 6$

$\therefore y = 1$

$x + y - z = 2 \Rightarrow x = 2 - y + z$

$= 2 - 1 - 2$

$= -1$

$\therefore$ solution is $x = -1$, $y = 1$, $z = -2$

HW (3)

$$\begin{bmatrix} 4 & 0 & 6 \\ -1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$$

HW (3)

$-2y - 2z = 8$

$3x + 4y - 5z = 8$

$$\tilde{A} = \left[\begin{array}{ccc:c} 0 & -2 & -2 & 8 \\ 3 & 4 & -5 & 8 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 3 & 4 & -5 & 8 \\ 0 & -2 & -2 & 8 \end{array}\right]$$

$$-2y-2z=8$$
$$y+z=-4$$
$$y=-4-z$$
$$3x+4y-5z=8$$
$$3x=8-4y+5z$$
$$=8-4(-4-z)+5z$$
$$=\cancel{8+16+4z+5z-z}$$
$$=8+16+4z+5z$$
$$=24+9z$$
$$x=8+3z$$

$\therefore$ $x=8+3t$, $y=-4-t$, $z=t$ infinite solution

L

UQ 2017 (4)

$$y+z-2w=0$$
$$2x-3y-3z+6w=2$$
$$4x+y+z-2w=4$$

$$\tilde{A}=\left[\begin{array}{cccc:c} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\rightarrow \left[\begin{array}{cccc:c} 4 & 1 & 1 & -2 & 4 \\ 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right]$$

$R_2 - \frac{1}{2} R_1$

$$\rightarrow \left[\begin{array}{cccc:c} 4 & 1 & 1 & -2 & 4 \\ 0 & -7/2 & -7/2 & 7 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right]$$

$\rightarrow \quad y + z - 2w = 0$

$y = 2w - z$

$-7/2\, y$

$R_3 + \frac{2}{7} R_2$

$-2 + \frac{2}{7} \times \frac{7}{2}$

$$\rightarrow \left[\begin{array}{cccc:c} 4 & 1 & 1 & -2 & 4 \\ 0 & -7/2 & -7/2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad -7/2\, y - 7/2\, z + 7w = 0$

$y + z - 2w = 0 \quad \Rightarrow \quad y = 2w - z$

$4x + y + z - 2w = 4 \Rightarrow 4x = 4 - (2w - z) + z - 2w$

$= \cancel{4 - 2w + z + z} - 2w$

$= \cancel{4 - 4w} + 2z \quad 4 \qquad x = 1$

~~x = 1 - w + z/2~~

$w = t_1, \quad z = t_2, \quad y = 2t_1 - t_2 \qquad x = 1$

Additional Problems

1. For what values of λ & μ the given system of equations

$x+y+z=1, \quad x+2y+3z=10, \quad x+2y+\lambda z=\mu$

has (a) no solution (b) a unique solution (c) infinite no. of solutions.

$x+y+z=1$
$x+2y+3z=10$
$x+2y+\lambda z=\mu$

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 9 \\ 0 & 1 & \lambda-1 & \mu-1 \end{array}\right] \quad \begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad R_3-R_2$$

(a) if $\lambda=3$ & $\mu=10$

rank $(\tilde{A})$ = rank $A = 2 < 3$ (no. of variables)

$\therefore$ the system has infinite number of solutions

(ii) if $\lambda = 3$ & $\mu \neq 0$, then $\rho(A) = 2$, $\rho(\tilde{A}) = 3$
$\therefore$ Hence system has no solution.

(iii) If $\lambda \neq 3$ & μ has any value then
Rank $(\tilde{A})$ = Rank $(A) = 3$ = no. of variables
system has a unique solution.

2. For what values of 'k' the system of equations
$x + y + z = 2$, $x + 2y + z = -2$, $x + y + (k-5)z = k$
has no solution.

3.

$x + y + z = 1$

$x + 2y + 3z = \mu$

$x + 5y + 9z = \mu^2$

$r(A) = r(AB) = 2$

$\mu^2 - 4\mu + 3 = 0$

$\mu = 1, 3$

find soln.

Additional Problem- owl book.

k-10
303

Tutorial

1) Solve $3x+2y+z=0$

$2x+3z=0$

$x+2y+3z=0$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

trivial soln. $x=0, y=0, z=0$

2) $x+3y+2z=0$

$2x-y+3z=0$

$3x-5y+4z=0$

$x+17y+4z=0$

$$[A:B] \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1/7 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1/7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x+3y+2z=0$

$y+1/7 z=0$

$y=-1/7 t$ & $z=-11/7 t$

1/7
2: Regular

Solve by Gauss ellimination

$y+z-2w=0$, $2x-3y-3z+6w=2$

$4x+y+z-2w=4$

$z=t_1$, $w=t_2$, $y=2t_2-t_1$, $x=2$

Qn)
$y+z-2w=0$
$2x-3y-3z+6w=2$
$4x+y+z-2w=4$

$$[A:B] = \left[\begin{array}{cccc:c} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right]$$

Qn. $x+y+z=1$, $x+2y+3z=\mu$, $x+5y+9z=\mu^2$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 4 & 8 & \mu^2-1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & \mu^2-4\mu+3 \end{array}\right]$$

$R_3 \to R_3 - 4R_2$

$\mu^2-1-4(\mu-1)$

$\mu^2-1-4\mu+4$

$-1, -3$

(1) $\mu^2-4\mu+3=0 \Rightarrow (\mu-1)(\mu-3)=0$

$\Rightarrow \mu=1, 3$

If $\mu=1, 3$ system is consistent &

$r[A:B] < n$ so infinite no of solution.

$\mu = 1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + y + z = 1$

$y + 2z = 0 \quad \Rightarrow \quad \cancel{x = -y - 2z = -t_1 - 2t_2}$

$y + 2z = 0 \Rightarrow \quad y = -2z \Rightarrow y = -2t_1$

$x = 1 - y - z \quad = 1 + 2t_1 - t_1 \quad = 1 + t_1$

$\mu = 3$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + y + z = 1$

$y + 2z = 2 \quad \Rightarrow \quad y = 2 - 2z \quad = 2 - 2t_1$

$\Rightarrow \quad x = 1 - y - z = 1 - (2 - 2t_1) - t_1$

$= -1 + t_1 = \underline{\underline{t_1 - 1}}$

2017
apply

(1) Find the rank of matrix A

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & 33/5 & 1/5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$r = 4$

'17

(2) Reduce to echelon form & find the rank.

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$r = 2$

'17
regular.

(3) Find ~~the~~ a basis for the null space

of $\begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 8 \\ 2 & 0 & 4 \end{bmatrix}$

$AX = 0$

$r = 2$

$$\sim \begin{bmatrix} 2 & -2 & 0 & : & 0 \\ 0 & 4 & 8 & : & \end{bmatrix}$$

$r = 2$

$(2, 2, -1)$

Matrix Eigen value Problems.

Let A be a given non zero square matrix of dimension $n \times n$. Consider the following vector equation $AX = \lambda X$, the problem of finding nonzero X's and λ's that satisfy the equation $AX = \lambda X$ is called eigen value problem.

A value of λ for which $AX = \lambda X$ has a solution $X \neq 0$ is called as Eigenvalue or charecteristic value of matrix A. The corresponding solutions $X \neq 0$ is called eigen vector of matrix A (or charecteristic vector)

Determination of Eigen values and Eigen vectors

Consider a system of equations,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n$$

gives,

$$(a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n = 0$$

in Matrix Notation,

$$(A - \lambda I) X = 0$$

Now this homogeneous system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficient is zero.

i.e, $D(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$

Charecteristic Matrix

The matrix $A - \lambda I$ is called charecteristic matrix

Charecteristic determinant

The determinant $\det(A - \lambda I) = D(\lambda)$ is called characteristic determinant.

Charecteristic equation

The equation $\det[A - \lambda I] = 0$ is called the charecteristic equation.

Charecteristic polynomial

The expansion of $D(\lambda)$ is a polynomial in λ of nth degree. which is called charecteristic polynomial.

Eigen value.

The eigen values of a square matrix A are the roots of the characteristic equation $\det(A - \lambda I) = 0$

Note: An $n \times n$ matrix has at least one eigen value and at most n numerically different eigen values.

* A & A^T have the same eigen value.

Eigen vector

Corresponding to each eigen value, the solution $X \neq 0$ of eqn $(A - \lambda I)X = 0$ is called eigen vector

Theorem:

If W and X are eigen vectors of a matrix A corresponding to the same eigen value λ so are $W + X$ (provided $~~X~~ \neq -W$) and kX for any $k \neq 0$

Eigen Space

The eigenvectors corresponding to one and the same eigen value λ of A, together with 0, form a vectorspace of called eigenspace of A corresponding to that λ.

Spectrum :

The set of all eigenvalues of a matrix A is called the spectrum of A.

Spectral radius :

The largest of the absolute values of the eigenvalues of A is called spectral radius of A.

Algebraic Multiplicity of e.v. λ

The order M_λ of an eigen~~vector~~ value λ as a root of ch. polynomial is called algebraic Multiplicity

Geometric Multiplicity

The number m_λ of L.I eigenvectors corresponding to λ is called geometric multiplicity of λ.
Thus M_λ is the dimension of the eigenspace corresponding to this λ.

Defect of λ

denoted as $\Delta_\lambda = M_\lambda - m_\lambda$

Ch. eqn. for 2nd order matrix

$$\lambda^2 - (\text{trace } A)\lambda + |A| = 0$$

Ch. eqn. for 3rd order matrix

$$\lambda^3 - (\text{trace} A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$$

A_{ii} is the det. of matrix by deleting ith row & column

Problems.

(2.) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

characteristic equation is given by

$$\lambda^3 - (tr\,A)\lambda^2 + (a_{11} + a_{22} + a_{33})\lambda - |A| = 0$$

$$tr\,A = -1$$

$$a_{11} = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} = -12$$

$$a_{22} = \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} = -3$$

$$a_{33} = \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -2 - 4 = -6$$

$$a_{11} + a_{22} + a_{33} = -12 - 3 - 6 = -21$$

$$|A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) + -3(-4+1)$$

$$= 24 + 12 + 9 = 45$$

$\therefore$ ch. eqn. is,

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$\therefore \lambda = 5, -3, -3$

let $\lambda_1 = 5, \quad \lambda_2 = -3, \quad \lambda_3 = -3 \qquad M_5 = 1 \quad M_{-3} = 2$

Eigen vector corresponding to $\lambda_1 = 5$ is given by,

$[A - 5I] X = 0$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply Gauss ellimination method,

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$

$\therefore$ we have $\quad -\frac{24}{7} x_2 + -\frac{48}{7} x_3 = 0$

$$-\frac{24}{7} x_2 = \frac{48}{7} x_3$$

$$x_2 = -2x_3$$

put $x_3 = 1$, so that $x_2 = -2$

also from, $-7x_1 + 2x_2 - 3x_3 = 0$

$$-7x_1 = -2x_2 + 3x_3$$
$$= 4 + 3 = 7$$
$$x_1 = -1$$

Hence the eigenvector corresponding to $\lambda = 5$ is

$$\begin{bmatrix} -1 \\ -2 \\ +1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = -3$

$$[A + 3I]X = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$n - r = 3 - 1 = 2$

rank = 1

$\rightarrow$ 2 L.I eiv

also, $x_1 + 2x_2 - 3x_3 = 0$

Then $x_1 = -2x_2 + 3x_3$

choosing $x_2 = 1,\ x_3 = 0$ and $x_2 = 0,\ x_3 = 1$

we obtain two linearly independent eigen vectors of A given by,

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$m_\lambda = 2$ for $\lambda = -3$

defect of λ

$\Delta_{-3} = 2 - 2 = 0$

I

Determine the eigenvalue & eigenvectors of

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

ch. eqn. is

$D(\lambda) = |A - \lambda I| = 0$

i.e. $\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$(-5-\lambda)(-2-\lambda) - 4 = 0$

i.e, $\lambda^2 + 7\lambda + 6 = 0$

$\therefore \lambda_1 = -1, \quad \lambda_2 = -6.$

Eigen vector corresponding to $\lambda_1 = -1$

$(A+I)X = 0$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\cancel{-4x_1}$ $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$

$-4x_1 + 2x_2 = 0$

$4x_1 = 2x_2 \Rightarrow x_1 = \frac{1}{2} x_2$ or $x_2 = 2x_1$

$r = 1, \quad n = 2, \quad$ Let $x_1 = 1$

$\Rightarrow x_2 = 2$

$\therefore$ eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

When $\lambda_2 = -6$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = t$
$x_2 =$

$x_2 = t$
$x_1 = 2t$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 + 2x_2 = 0 \quad x_1 = -2x_2$

$x_2 = 1, \quad x_1 = -2$

$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(3) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\lambda^2 - (\text{trace}A)\lambda + |A| = 0$

$\lambda^2 - 0\lambda + -1\times 0 = 0$

$\lambda^2 = 0$, ~~$\lambda = \pm 1$~~

$\lambda = 0$ eigen value of algebraic multiplicity

$M_0 = 2$

Now corresponding eigen vector,

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = 0 \Rightarrow x_1$ is any value

hence $m_0 = 1$
defect $\Delta_0 = 1$

eig. vector $\underline{\underline{[1 \quad 0]^T}}$

---

(4) $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$ $\lambda = 3$ eig value with $M_3 = 2$

eigen vector,

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$2x_2 = 0 \Rightarrow x_2 = 0.$ $\quad x_2 =$ any value.

$\therefore$ eigen vector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore m_3 = 1$

(5) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$

when $\lambda = +i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-ix_1 + x_2 = 0 \qquad x_2 =$

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-ix_1 + x_2 = 0 \qquad x_2 = ix_1$

when $x_1 = 1, \; x_2 = i$

$X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Illy for $\lambda = -i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i x_1 + x_2 = 0$$

$$i x_1 = -x_2$$

$x_1 = 1, \quad x_2 = -i$ $\qquad \Delta i = 1-1 = 0$

$\therefore X_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\qquad \Delta -i = 1-1 = 0$

Exercise

hw 1 $A = \begin{bmatrix} 3/2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\lambda^2 - 9/2\,\lambda + 9/2 = 0$$

$$\lambda = 3/2,\ 3$$

Eigvector for $\lambda = 3/2 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3 \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 3 & -2 \\ 9 & -6 \end{bmatrix}$ $\qquad \lambda^2 + 3\lambda = 0, \qquad \lambda = 0, -3$

$\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

(3) $\begin{bmatrix} 0.20 & -0.40 \\ 0.40 & 0.20 \end{bmatrix}$

$\lambda^2 - 0.40\lambda + 0.2 = 0$

$\lambda = .2 \pm .4i$

Eig vector for $\lambda = .2 + .4i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$

Eig vector for $\lambda = .2 - .4i$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

---

(3) $\begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$\text{tr } A = 12, \quad A_{11} = 15, \quad A_{22} = 8 \quad A_{33} = 16$

$|A| = 60 - 12 - 20 = 28$

ch. eqn. is $\lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$

$(\lambda - 4)(\lambda - 7)(\lambda - 1) = 0$

$\lambda = 1, 4, 7.$

Eig. vector for $\lambda = 1$

$$\begin{bmatrix} 4-1 & 2 & -2 \\ 2 & 5-1 & 0 \\ -2 & 0 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -2 \\ 0 & 8/3 & 4/3 \\ 0 & 4/3 & 2/3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2/3 R_1 \\ R_3 + 2/3 R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 2 & -2 \\ 0 & 8/3 & 4/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 - 2R_2$$

$r = 2$, dim $= 1$

$$8/3\, x_2 + 4/3\, x_3 = 0$$

$$\Rightarrow 8/3\, x_2 = -4/3\, x_3$$

$$\Rightarrow x_2 = -1/2\, x_3$$

also $3x_1 + 2x_2 - 2x_3 = 0$

$$\therefore x_1 = \frac{1}{3}\left[-2x_2 + 2x_3\right]$$

$$= x_3$$

$\therefore$ if $x_3 = 1$, $x_1 = 1$, $x_2 = -1/2$ Eig. vector is $\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$

<u>Eig. vector for $\lambda = 4$.</u>

Consider $[A - \lambda I] = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ -2 & 0 & -1 \end{bmatrix}$ $\quad R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad R_3 \to R_3 + R_1$

$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad R_3 \to R_3 - \frac{1}{2} R_2$

$r = 2.$ $\quad 2x_2 - 2x_3 = 0$

$x_2 = x_3.$

$2x_1 + x_2 = 0$

$2x_1 = x_2 = x_3$

$x_1 = \frac{1}{2} x_3$

let $x_3 = 1$ $\therefore x_1 = \frac{1}{2}$, $x_2 = 1$

eg. vector is $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ //

Eigen vector for $\lambda = 7$

$$[A - 7I] = \begin{bmatrix} -3 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & 2 & -2 \\ 0 & -2/3 & -4/3 \\ 0 & -4/3 & -8/3 \end{bmatrix} \quad \begin{matrix} R_2 + 2/3 R_1 \\ R_3 - 2/3 R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -3 & 2 & -2 \\ 0 & -2/3 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad -2/3\, x_2 - 4/3\, x_3 = 0$$

$$x_2 = -2x_3$$

$$-3x_1 + 2x_2 - 2x_3 = 0$$

$$-3x_1 = 2x_3 - 2x_2 = 2x_3 + 4x_3$$

$$= 6x_3$$

$$x_1 = -2x_3$$

$x_3 = 1, \quad x_1 = -2, \quad x_2 = -2$

$\therefore$ eig. vector is $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

trace $A = 8$

$A_{11} = 4, \quad A_{22} = 3, \quad A_{33} = 12$

$|A| = 3(3 \times 4) - (5 \times 0) + 3(0) = 12$

$\therefore \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

$\lambda = 1, 3, 4$

<u>Eig. vector for $\lambda = 1$</u>

$[A - I] = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad r = 2$

$3x_2 + 6x_3 = 0 \implies 3x_2 = -6x_3$

$\implies x_2 = -2x_3$

$2x_1 + 5x_2 + 3x_3 = 0 \implies 2x_1 = -5x_2 - 3x_3$

$= +10x_3 - 3x_3 = +7x_3$

$x_1 = -\cancel{13}\,7/2\; x_3$

$x_3 = 1, \quad x_2 = 2, \quad x_1 = \cancel{-13/2}\; 7/2$

$\boxed{\begin{bmatrix} 7/2 \\ -2 \end{bmatrix}}$

Eigen vector for $\lambda = 3$

$$[A-3I] = \begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$6 - 3/5$

$$\sim \begin{bmatrix} 0 & 5 & 3 \\ 0 & 0 & 27/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{30-3}{5}$

$$\sim \begin{bmatrix} 0 & 5 & 3 \\ 0 & 0 & 27/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad \frac{27}{5} x_3 = 0 \quad$ also $\quad 5x_2 + 3x_3 = 0$

$x_3 = 0 \qquad 5x_2 = -3x_3$

$x_2 = 0$

$\therefore x_2 = 0$ & $x_3 = 0$ hence x_1 can take any value

$\therefore \quad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Eigen vector for $\lambda = 4$

$$[A-4I] = \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$6x_3 = 0$

$x_3 = 0$

$-x_1 + 5x_2 + 3x_3 = 0$

$-x_1 = -5x_2$

$x_1 = 5x_2$

$x_2 = 1, \quad x_1 = 5. \quad x_3 = 0$

$\therefore \quad X = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

Qn. Algebraic & geometric multiplicity defect.

Eigen space

Spectrum.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

24 Sunday

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$(\lambda - 2)^3 = 0$

$\lambda = 2, 2, 2$

Eig vector for $\lambda = 2$

$\lambda = 2, 2, 3$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Determine algebraic multiplicity, geometric multiplicity and defect of $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 0 & -3 & 4 \end{bmatrix}$ $\lambda = -2, 1, 1$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ $x_3 = 0$, x_1 & x_2 can have any value

$\therefore x_1 = 0, \; x_2 = 1 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$x_1 = 1, \; x_2 = 0 \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

HW $\begin{bmatrix} 6 & 5 & 2 \\ 2 & 0 & -8 \\ 5 & 4 & 0 \end{bmatrix}$ HW (1) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ $\lambda = 1, 1, 5$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} a & 1 \\ -k & a \end{bmatrix}$ $\lambda = 2, 2, 2$ $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(2) HW $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $\lambda = -2, 3, 6$ $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

find a & b s.t. 3 & -2 are eigen values

$\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$

Symmetric, Skew symmetric & Orthogonal Matrices.

A real square matrix $A = [a_{jk}]$ is called symmetric if,

$$A^T = A \quad \text{thus} \quad a_{kj} = a_{jk}.$$

Skew symmetric if,

$$A^T = -A, \quad \text{then} \quad a_{kj} = -a_{jk}$$

Orthogonal if,

$$A^T = A^{-1}$$

$$AA^T = A^TA = I$$

$$AA^{-1} = A^{-1}A = I$$

Note:

Any real square matrix A may be written as the sum of a symmetric matrix R and a skew symmetric matrix S. where,

$$R = \frac{1}{2}(A + A^T) \quad \& \quad S = \frac{1}{2}(A - A^T)$$

Theorem

(a) The eigen values of a symmetric matrix are real.

(b) The eigen values of a skew symmetric matrix are pure imaginary or zero.

Note:

1) Determinant of an orthogonal matrix has value ± 1

2) The diagonal elements of a skew symmetric matrix is zero.

(1) Prove that the matrix $\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ is symmetric.

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$A = A^T$$

$\therefore$ Symmetric.

---

(2) Is the matrix $\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ skew symmetric.

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ -12 & -20 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & -9 & -12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 9 & 12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} = -A$$

$\therefore$ A is skew symmetric.

Properties of Eigen values and Eigen vectors

1) The trace of the matrix is the sum of the eigenvalues of a matrix.

2) The determinant of the matrix A equals the product of eigenvalues of a matrix.

3 If $\lambda_1, \lambda_2, \dots \lambda_n$ are n eigenvalues of A then
 (i) eigen values of kA are $k\lambda_1, k\lambda_2 \dots k\lambda_n$
 (ii) eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots \frac{1}{\lambda_n}$
 (iii) eigen values of A^m are $\lambda_1^m, \lambda_2^m \dots \lambda_n^m$

4. Eigenvalues of a symmetric matrix are real

5. Eigenvalues of a skew symmetric matrix are purely imaginary or zero.

6. The eigenvalues of a square matrix and its transpose are same.

7. Zero is the eigenvalue of a matrix iff the matrix is singular.

8. The eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

9 If λ is an eigenvalue of A, then $\lambda - k$ is an eigen value of $A - kI$

10) If λ is an eigenvalue of A then $\frac{|A|}{\lambda}$ is

(9) If A & B are 2 square invertible matrices then AB & BA have the same eigenvalue.

(10) Matrices $A^{-1}B$ & BA^{-1} have the same eigenvalue (A is invertible)

(11) $P^{-1}AP$ have the same eigenvalues as A if P is an invertible matrix.

Theorem

The diagonal elements of a skew symmetric matrix must be zero

Proof

Since $A = [a_{ij}]$ is skew symmetric.

$$A^T = -A \quad \text{ie,} \quad a_{ij} = -a_{ji}$$

$$\therefore \quad a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$$

$\therefore$ diagonal elements are zeros.

(3) $\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$ P.T matrix is orthogonal

To prove $A^T = A^{-1}$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$|A| = 2/3\left[\frac{-4}{3} - 2/3\right] - 1/3\left[4/3 - 1/3\right] + 2/3\left[-4/3 - 2/3\right]$$

$$= \frac{2}{3} \times \frac{-6}{3} - 1/3 \times 3/3 + 2/3 \times -6/3$$

$$= \frac{-4}{3} - \frac{1}{3} - 4/3 = -9/3 = -3$$

$$\text{adj } A = (\text{Cofact } A)^T$$

$$\text{Cofactor } A = \begin{bmatrix} +2 & -1 & +2 \\ +2 & -2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \quad \text{Cofac } A^T = \begin{bmatrix} 2 & 2 & -1 \\ -1 & -2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$$

$$* \; A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} -2/3 & -2/3 & +1/3 \\ +1/3 & +2/3 & +2/3 \\ -2/3 & +1/3 & -1/3 \end{bmatrix}$$

Exercise

1. $\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$ Are the following matrices symmetric, skew symmetric or orthogonal. Find the spectrum of each.

(i) $A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$ $A^T = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$

not symmetric, not skew symmetric.

$|A| = 9/25 + 16/25 = 25/25 = 1$

$A^{-1} = \dfrac{\text{adj}A}{|A|}$ $\begin{bmatrix} 3/5 & +4/5 \\ 4/5 & -3/5 \end{bmatrix}$

$= \begin{bmatrix} 3/5 & +4/5 \\ -4/5 & +3/5 \end{bmatrix} = A^T$

Orthogonal

Eigen values

ch eqn.

(3)

$$\lambda^2 - \frac{6}{5}\lambda + 1 = 0$$

$$\frac{\frac{6}{5} \pm \sqrt{\frac{36}{25} - 4}}{2}$$

$$\frac{\frac{6}{5} \pm \sqrt{\frac{-64}{25}}}{2}$$

$$= \frac{\frac{6}{5} \pm \frac{8}{5}i}{2}$$

$$= \frac{14}{10}i, \frac{-2}{10}i$$

$$\lambda = \frac{7}{5}i, \frac{-1}{5}i$$

Spectrum $= \left\{\frac{7}{5}i, -\frac{1}{5}i\right\}$

(2) $A = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$, $|A| = 1 + 16 = 17$

$\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} / 17$

not orthogonal, not skew symmetric.

eig. value, $1 \pm 4i$

(3) $\begin{bmatrix} 0 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1/2 & -1 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 1 & 1/2 \\ -1 & 0 & -1 \\ -1/2 & 1 & 0 \end{bmatrix}$

$A^T = -A$, skew symmetric.

non orthogonal.

eig values, $0 \pm 3/2\, i$

(4) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$|A| = 1$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\quad \frac{\text{adj } A}{|A|} = \begin{bmatrix} \cos\theta & +\sin\theta \\ \mp\sin\theta & \cos\theta \end{bmatrix}$

$= \underline{\underline{A^T}}$

orthogonal.

ch. eqn.

$\lambda^2 - (2\cos\theta)\lambda + 1 = 0$

$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$

$= \underline{\underline{\cos\theta \pm i\sin\theta}}$

Spectrum $= \{\cos\theta + i\sin\theta, \; \cos\theta - i\sin\theta\}$

---

(5) $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Cofactor matrix of A $= \begin{bmatrix} 0 & 0 & +1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$|A| = 0 - 0 + 1(-1) = {}^{-}{+1}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} 0 & 0 & ^{+}\mp 1 \\ 0 & \mp 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\lambda^3 + \lambda^2 + (0 + 1 + 0)\lambda - 1 = 0$$

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

$$\lambda = -1, \pm i$$

Spectrum, $= \{-1, \pm i\}$

Diagonalisation of a matrix

If an $n \times n$ matrix A has a basis of eigenvectors then $D = \overset{P^{-1}AP}{X^{-1}AX}$ is diagonal with the eigen values of A as the entries on the main diagonal. Here X is the matrix with these eigenvectors as column vectors.

Also,

$$D^m = X^{-1}A^mX. \qquad m = 2, 3, \ldots$$

$$A^m = PD^mP^{-1}$$

$\begin{cases} D - \text{spectral Matrix.} \\ P - \text{modal matrix.} \end{cases}$

1. Diagonalise $A = \begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix}$

$$\boxed{A^{-1} = PD^{-1}P^{-1}}$$

ch. eqn,

$$\lambda^3 - (\text{trace } A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$$

trace $A = 6 + 2 + 9 = 17$

$A_{11} = 18, \quad A_{22} = 54, \quad A_{33} = 12$

$|A| = 6 \times 18 = 108$

$\therefore \quad \lambda^3 - 17\lambda^2 + 84\lambda - 108 = 0$

$\lambda = 2$

$$(\lambda - 2)(\lambda^2 - 15\lambda + 54) = 0$$

$$(\lambda - 2)(\lambda - 9)(\lambda - 6) = 0$$

$$\lambda = 2, 9, 6$$

$\underline{\lambda = 2}$

$$\begin{bmatrix} 4 & 0 & 0 \\ 12 & 0 & 0 \\ 21 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -6 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & -6 & 7 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 0 \\ 7/6 \\ 1 \end{bmatrix}$$

$4x_1 = 0, \qquad -6x_2 + 7x_3 = 0$

$x_1 = 0 \qquad -6x_2 = -7x_3 \qquad x_2 = 7/6\, x_3$

$x_3 = 1$

$\underline{\lambda = 9}$

$$\begin{bmatrix} -3 & 0 & 0 \\ 12 & -7 & 0 \\ 21 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & -6 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0, \quad x_2 = 0, \quad x_3 = 1$

$\underline{\lambda = 6}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 12 & -4 & 0 \\ 21 & -6 & 3 \end{bmatrix} \sim \begin{bmatrix} \cancel{12} & \cancel{-4} & \cancel{0} \\ 21 & -6 & 3 \\ 12 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$12 - \frac{12 \times 21}{21}$

$-4 + \frac{12}{21} \times 6$

$\frac{-84 + 72}{21}, \quad \frac{-4}{7} \quad \frac{-12}{21}$

$$\sim \begin{bmatrix} 21 & -6 & 3 \\ 0 & -4/7 & -12/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$-4/7\, x_2 = 0$

$x_2 = 0$

$21x_1 + 3x_3 = 0 \qquad 21x_1 = -3x_3$

$x_1 = \frac{-1}{7} x_3$

$x_3 = 1, \quad x_1 = -1/7$

$\lambda = 6$

$$-\frac{4}{7}x_2 + \frac{\pm 12}{7}x_3 = 0 \Rightarrow x_2 = \frac{-12}{7} \times \frac{7}{4} x_3$$
$$= -3x_3$$

also $21x_1 = 6x_2 - 3x_3$

$x_1 = -x_3$ $\therefore X_3 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

Let

$$X = \begin{bmatrix} 0 & -1 & 0 \\ 7/6 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$ $D = P^{-1}AP$

$= D$

(3)

(2) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$\lambda^2 - 4\lambda - 12 = 0$

$\lambda = -2, 6$

for $\lambda = -2$

eig vector $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

for $\lambda = 6$

eig vector $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \cancel{X} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$P^{-1}AP = X^{-1}AX = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}$

$= D$

(3) $\begin{bmatrix} -1 & 2 & -2 \\ 2 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

$\lambda = 5, -1, 3$

$X = \begin{bmatrix} 0 & -3 & -1/2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(HW) Qn. $\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\lambda = 1, ..$

$X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $P = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1 \end{bmatrix}$

(4) $\begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$

$\lambda = 3, -4, 0$

$X = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Qn) $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$

$\lambda = -2, 1, 1$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(5) $A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ $\lambda = 3, 3$ $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Not diagonalizable

Quadratic forms

A quadratic form in two variables x_1, x_2 is an expression of the form $c_{11}x_1^2 + c_{12}x_1x_2 + c_{22}x_2^2$ where each term has degree 2.

Ily quadratic form in 3 variables is given by

$$c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + c_{22}x_2^2 + c_{23}x_2x_3 + c_{33}x_3^2$$

Matrix of quadratic form in 2 variables

Consider, $c_{11}x_1^2 + c_{12}x_1x_2 + c_{22}x_2^2$

Consider Matrix $A = \begin{bmatrix} c_{11} & c_{12}/2 \\ c_{12}/2 & c_{22} \end{bmatrix}$ or

$$= \begin{bmatrix} \text{Coeff } x_1^2 & \tfrac{1}{2}\text{Coeff } x_1x_2 \\ \tfrac{1}{2}\text{Coeff } x_2x_1 & \text{Coeff. } x_2^2 \end{bmatrix}$$

Then we have,

$$Q = X^T A X = c_{11}x_1^2 + c_{12}x_1x_2 + c_{22}x_2^2$$

where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Matrix A is called coefficient matrix of the quadratic form.

A will be a symmetric matrix.

Consider the matrix X^* obtained by writing the normalised eigenvectors as the column vectors. Normalised form is obtained by dividing each component of the vector by the square root of sum of squares of all components. Then X^* will be a orthogonal matrix. i.e. ${X^*}^T = {X^*}^{-1}$ (Since by the theorem symmetric matrix has an orthonormal basis of eigen vectors)

Principal Axes Theorem

The orthogonal transformation $X = X^* Y$ (where X^* is the orthogonal matrix) transforms a quadratic form $Q = X^T A X$ to the principal axes form or canonical form,

$$Q = Y^T D Y$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

$Y^T D Y$ where D is a diag matrix by the orthogonal transformation $X = PY$ where P is an orthogonal matrix which is obtained by grouping normalised eig. vectors as column vectors.

$X = PY, \quad Q = X'AX =$

$= Y^T(P^T AP)Y = Y^T(P^{-1}A$

$= Y^T DY = \lambda_1 y_1^2 + \lambda_2$

where $\lambda_1, \lambda_2 \ldots \lambda_n$ are the (not eigen values of the matrix A and orthogonal matrix with corresponding ... vectors $x_1, x_2 \ldots x_n$ respectively as column vectors

---

(1) What kind of conic section is given by the quadratic form. Transform it into principal axes. Express $X^T = [x_1 \ x_2]$ in terms of the new co-ordinate vector $Y^T = [y_1 \ y_2]$

$Q = 17x_1^2 - 30x_1 x_2 + 17x_2^2 = 128$

The principal axes form is $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

coeff. matrix A of quadratic form is

$$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

ch eqn. $\lambda^2 - 34\lambda + 64 = 0$

$(\lambda - 32)(\lambda - 2) = 0$

$\lambda = 32, 2$

$\therefore$ principal axes form is $2y_1^2 + 32y_2^2$

To find conic section:

given $Q = 128$

$\Rightarrow 2y_1 + 32y_2^2 = 128$

$\Rightarrow \dfrac{y_1^2}{64} + \dfrac{y_2^2}{4} = 1$

$\Rightarrow \dfrac{y_1^2}{8^2} + \dfrac{y_2^2}{2^2} = 1$ which is an ellipse

To find the transformation

$X = X^* Y$

Eigen vector for $\lambda = 2$

$(A - 2I) \sim \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \sim \begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix}$

$15x_1 - 15x_2 = 0 \qquad x_1 = x_2$

eig vector $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Normalised form $= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Eigen vector for $\lambda = 32$

$$(A - 32I) = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \sim \begin{bmatrix} -15 & -15 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$ eig. vector $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

normalised form $= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Now, $X^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X^* Y = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

transforms quadratic form into canonical form.

(2) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

Coeff Matrix $A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$

ch. eqn. $\lambda^2 - 14\lambda + 40 = 0$

$\lambda = 10, 4$

$\therefore$ principal axes form is

$$Q = 10y_1^2 + 4y_2^2$$

Conic Section:

$$Q = 200$$

$$10y_1^2 + 4y_2^2 = 200$$

$$\frac{y_1^2}{20} + \frac{y_2^2}{50} = 1$$

$\Rightarrow$ ellipse

Orthogonal transformation

$$x = X^* y$$

for $\lambda = 10$

$$(A - 10I) = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 \qquad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda = 4$

$$[A - 4I] = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \qquad x_1 = -x_2$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore X^* = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$X = X^* Y = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

---

(3) $3x_1^2 + 22x_2x_1 + 3x_2^2 = 0$

$$A = \begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix}$$

ch. eqn. $\lambda^2 - 6\lambda - 112 = 0$

$$\lambda = 14, -8$$

$$Q = 14y_1^2 - 8y_2^2$$

Conic section:

$$Q = 0$$

$$14y_1^2 - 8y_2^2 = 0$$

$$\Rightarrow y_1^2 = \frac{8}{14} y_2^2$$

$$\Rightarrow y_1 = \pm \sqrt{\frac{8}{14}}\, y_2$$ pair of straight lines.

eigen vectors for $\lambda = 14$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eg. vector for $\lambda = -8$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore X^* = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore X = X^* Y$$

(ii) $x_1^2 - 12 x_1 x_2 + x_2^2 = 70$

$$A = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix}$$

$$\lambda^2 - 2\lambda - 35 = 0$$

$$\lambda = 7, -5$$

$7y_1^2 - 5y_2^2 = Q$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $X^* = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Conic section - hyperbola

5) $-11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

$$A = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix}$$

$$\lambda^2 - 13\lambda - 2028 = 0$$

$$\lambda = 52, -39$$

$$Q = 52y_1^2 - 39y_2^2$$

eig vector for $\lambda = 52$

$$[A - 52I] = \begin{bmatrix} -63 & 42 \\ 42 & -28 \end{bmatrix}$$

$$\sim \begin{bmatrix} -63 & 42 \\ 0 & 0 \end{bmatrix}$$

$\frac{42}{63} = \frac{6}{9} = \frac{2}{3}$

$$-63x_1 = -42x_2$$

$$x_1 = \frac{42}{63}x_2 = \frac{2}{3}x_2$$

$$X_1 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \text{ or } X_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

eig vector for $\lambda = -39$

$$[A + 39I] = \begin{bmatrix} 28 & 42 \\ 42 & 63 \end{bmatrix} \sim \begin{bmatrix} 28 & 42 \\ 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{-42}{28}x_2 = -\frac{3}{2}x_2 \quad X_2 = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\therefore X^* = \begin{bmatrix} \frac{2}{\sqrt{13}} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$$

$$\underline{\underline{X = X^* Y}}$$

Conic section

$$Q = 156$$

$$52y_1^2 - 39y_2^2 = 156$$

$$\frac{y_1^2}{3} - \frac{y_2^2}{4} = 1$$

hyperbola.

__Nature of Q.F__

Q.F → +ve definite all $\lambda > 0$

+ve semi definite $\lambda \geq 0$

−ve definite all $\lambda < 0$

−ve semi definite $\lambda \leq 0$

indefinite λ both +ve & −ve.

Sig nature = { no of zero eigen value, +ve eigen value, −ve eigen value }

Q1.(1) $A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ $\lambda = 3, 3$ not diagonalisable

(2) $\begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$ $\lambda = 3, -4, 0$

$$X = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz$$

S.T $Q > 0$ for all non zero values of x, y, z.

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & \text{coeff } x^2 & \tfrac{1}{2}xy & \tfrac{1}{2}xz \\ y & \tfrac{1}{2}yx & \text{coeff } y^2 & \tfrac{1}{2}yz \\ z & \tfrac{1}{2}xz & \tfrac{1}{2}yz & \text{coeff } z^2 \end{array}$$

05 Sunday

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Check whether symmetric, orthogonal
find eig. value & eig vectors.

n) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\lambda^2 - 2\cos\theta\,\lambda + 1 = 0$$

$$\lambda = \cos\theta \pm i\sin\theta$$

$\lambda = \cos\theta + i\sin\theta.$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-i\sin\theta + i\sin\theta = 0$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = t$

$-i\sin\theta(x) - \sin\theta(y) = 0$

$x = 1$

$y = -i$

$-i\sin\theta - \sin\theta\, y = 0$

$\sin\theta\, y = -i\sin\theta$

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$y = \underline{\underline{-i}}$

quadratic form involving 3 variables

$$ax^2 + by^2 + cz^2 + dxy + eyz + fxz$$

Matrix, $A = \begin{bmatrix} \text{Coeff. of } x^2 & \frac{1}{2}\text{Coeff } xy & \frac{1}{2}\text{Coeff } xz \\ \frac{1}{2}\text{Coeff of } yx & \text{Coeff } y^2 & \frac{1}{2}\text{coeff. of } yz \\ \frac{1}{2}\text{Coeff of } zx & \frac{1}{2}\text{Coeff } zy & \text{Coeff } z^2 \end{bmatrix}$

eg: $3x^2 + 4y^2 + 5z^2 + 2xy + 4yz + 6xz$

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$